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θ-Summation and Hardy Spaces[1]

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A general summability method of Fourier series and Fourier transforms is given with the help of an integrable function θ having integrable Fourier transform. Under some weak conditions on θ we show that the maximal operator of the θ-means of a distribution is bounded from $H_p(\mathbf{T})$ to $L_p(\mathbf{T})$ $(p_0<p<\infty)$ and is of weak type (1,1), where $H_p(\mathbf{T})$ is the classical Hardy space and $p_0<1$ is depending only on θ. As a consequence we obtain that the θ-means of a function $f \in L_1(\mathbf{T})$ converge a.e. to f. For the endpoint p_0 we get that the maximal operator is of weak type $(H_{p_0}(\mathbf{T}), L_{p_0}(\mathbf{T}))$. Moreover, we prove that the θ-means are uniformly bounded on the spaces $H_p(\mathbf{T})$ whenever $p_0<p<\infty$ and are uniformly of weak type $(H_{p_0}(\mathbf{T}), H_{p_0}(\mathbf{T}))$. Thus, in the case $f \in H_p(\mathbf{T})$, the θ-means converge to f in $H_p(\mathbf{T})$ norm $(p_0<p<\infty)$. The same results are proved for the conjugate θ-means and for Fourier transforms, too. Some special cases of the θ-summation are considered, such as the Weierstrass, Picar, Bessel, Fejér, Riemann, de La Vallée-Poussin, Rogosinski and Riesz summations.

Key Words: Hardy spaces; p-atom; atomic decomposition; interpolation; Fourier transforms; θ-summation.

1. INTRODUCTION

The Hardy–Lorentz spaces $H_{p,q}(\mathbf{T})$ of distributions are introduced with the $L_{p,q}(\mathbf{T})$ Lorentz norm of the non-tangential maximal function. Of course, $H_p(\mathbf{T}) = H_{p,p}(\mathbf{T})$ are the usual Hardy spaces $(0<p\leqslant\infty)$.

Butzer and Nessel [3] and recently Bokor, Schipp, Szili and Vértesi [2, 11, 12, 16, 17] considered a general method of summation, the so-called θ-summability. The θ-means of Fourier transforms can be written in a

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natural way as a singular integral of the Fourier transform of θ, $\hat{\theta}$ (see Butzer and Nessel [3]). They proved that if $\hat{\theta}$ can be estimated by a non-increasing integrable function, then the θ-means of a function $f \in L_1(\mathbf{R})$ converge a.e. to f. This convergence result is also proved there for the θ-means of Fourier series. As special cases they considered the Weierstrass, Picar, Bessel, Fejér, de La Vallée-Poussin and Riesz summations. For example, they verified that the Riesz means $\sigma_T^{\alpha,\gamma} f$ converge a.e. to f as $T \to \infty$ if $f \in L_1(\mathbf{R})$ and $\gamma = 1, 2$ (see also Stein and Weiss [14]).

The author [21] generalized this last result and proved that the maximal Riesz operator $\sigma_*^{\alpha,\gamma} := \sup_{T>0} |\sigma_T^{\alpha,\gamma}|$ is bounded from $H_p(\mathbf{R})$ to $L_p(\mathbf{R})$ provided that $0 < \alpha < \infty$, $1 \leqslant \gamma < \infty$, $1/(\min(\alpha, 1)+1) < p < \infty$ and, moreover, it is of weak type $(1, 1)$, i.e.

$$\sup_{\rho>0} \rho\lambda(\sigma_*^{\alpha,\gamma} f > \rho) \leqslant C \, \|f\|_1 \qquad (f \in L_1(\mathbf{R}))$$

(this last result for $\alpha = \gamma = 1$ can also be found in Zygmund [23] and Móricz [10]). This weak type inequality assures already the a.e. convergence of the Riesz means mentioned above.

In this paper we generalize these results. First we consider the θ-means of Fourier series and prove that the θ-means $U_n^\theta f$ of a function $f \in L_1(\mathbf{T})$ can be written also as a singular integral of f and $\hat{\theta}$ over $\mathbf{R}$. We introduce the maximal operator $U_*^\theta := \sup_{n\in\mathbf{N}} |U_n^\theta|$, the conjugate distribution $\tilde{f}$, the conjugate θ-means $\tilde{U}_n^\theta f$ and the conjugate maximal operator $\tilde{U}_*^\theta$.

Under some weak conditions on θ and $\hat{\theta}$ we will show that the maximal operators U_*^θ and $\tilde{U}_*^\theta$ are bounded from $H_{p,q}(\mathbf{T})$ to $L_{p,q}(\mathbf{T})$ whenever $p_0 < p < \infty$, $0 < q \leqslant \infty$ and are of weak type $(1, 1)$. The parameter p_0 is less than 1 and depending on θ. For this endpoint we can verify that the preceding two maximal operators are of weak type $(H_{p_0}(\mathbf{T}), L_{p_0}(\mathbf{T}))$.

A usual density argument implies then that $U_n^\theta f \to f$ a.e. and $\tilde{U}_n^\theta f \to \tilde{f}$ a.e. as $n \to \infty$, provided that $f \in L_1(\mathbf{T})$. Note that $\tilde{f}$ is not necessarily integrable whenever f is.

We will prove also that the operators U_n^θ and $\tilde{U}_n^\theta$ $(n \in \mathbf{N})$ are uniformly bounded in n from $H_{p,q}(\mathbf{T})$ to $H_{p,q}(\mathbf{T})$ $(p_0 < p < \infty,\ 0 < q \leqslant \infty)$ and are uniformly of weak type $(H_{p_0}(\mathbf{T}), H_{p_0}(\mathbf{T}))$. From this it follows that $U_n^\theta f \to f$ and $\tilde{U}_n^\theta f \to \tilde{f}$ in $H_{p,q}(\mathbf{T})$ norm (resp. in weak $H_{p_0}(\mathbf{T})$ norm) as $n \to \infty$, whenever $f \in H_{p,q}(\mathbf{T})$ $(p_0 < p < \infty,\ 0 < q \leqslant \infty)$ (resp. $f \in H_{p_0}(\mathbf{T})$).

As special case we investigate ten well known summability methods, amongst others the summations mentioned above.

We consider also the θ-means of Fourier transforms on the real line and prove all the results above in this context.

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2. HARDY SPACES AND CONJUGATE FUNCTIONS

Let $\mathbf{N}$ denote the none-negative integers, $\mathbf{R}$ the real numbers; $\mathbf{R}_+$ the positive real numbers, $\mathbf{T} := [-\pi, \pi)$ and λ be the Lebesgue measure. We also use the notation $|I|$ for the Lebesgue measure of the set I. We briefly write $L_{p,q}(\mathbf{X})$ instead of the real Lorentz space $L_{p,q}(\mathbf{X}, \lambda)$ $(0<p,\ q \leqslant \infty)$ and its norm is denoted by $\|\cdot\|_{p,q}$ where $\mathbf{X} = \mathbf{T}$ or $\mathbf{R}$ (for the exact definitions see e.g. Weisz [21] and the references there). We extend all functions on $\mathbf{T}$ periodically to $\mathbf{R}$.

Let f be a distribution on $C^\infty(\mathbf{T})$. The nth Fourier coefficient is defined by $\hat{f}(n) := f(e^{-\imath nx})$ where $\imath = \sqrt{-1}$. In special case, if f is an integrable function then

$$\hat{f}(n) = \frac{1}{2\pi} \int_{\mathbf{T}} f(x)\, e^{-\imath nx}\, dx \qquad (n \in \mathbf{N}).$$

The non-tangential maximal function of a distribution f is defined by

$$f^*(x) := \sup_{0<r<1} |f * P_r(x)|,$$

where $*$ denotes the convolution and

$$P_r(x) := \sum_{k=-\infty}^{\infty} r^{|k|} e^{\imath kx} = \frac{1-r^2}{1+r^2-2r\cos x} \qquad (x \in \mathbf{T})$$

is the Poisson kernel.

For $0<p,\ q \leqslant \infty$ the *Hardy–Lorentz space* $H_{p,q}(\mathbf{T})$ consists of all distributions f for which

$$\|f\|_{H_{p,q}(\mathbf{T})} := \|f^*\|_{p,q} < \infty.$$

Note that in case $p = q$ the usual definition of Hardy spaces $H_{p,p}(\mathbf{T}) = H_p(\mathbf{T})$ is obtained. For other equivalent definitions we call for Fefferman and Stein [5] and Stein [15]. Recall that $L_1(\mathbf{T}) \subset H_{1,\infty}(\mathbf{T})$, more exactly,

$$\|f\|_{H_{1,\infty}(\mathbf{T})} = \sup_{\rho>0} \rho\lambda(f^* > \rho) \leqslant \|f\|_1 \qquad (f \in L_1(\mathbf{T})). \tag{1}$$

Moreover,

$$H_{p,q}(\mathbf{T}) \sim L_{p,q}(\mathbf{T}) \qquad (1<p<\infty, 0<q \leqslant \infty), \tag{2}$$

where $\sim$ denotes the equivalence of the norms and spaces (see Fefferman and Stein [5], Stein [15], Fefferman, Riviere, Sagher [4]).

The following interpolation result concerning Hardy–Lorentz spaces will be used several times in this paper (see Fefferman, Riviere, Sagher [4] and also Weisz [19]).

THEOREM A. *If a sublinear (resp. linear) operator V is bounded from $H_{p_0}(\mathbf{T})$ to $L_{p_0}(\mathbf{T})$ (resp. to $H_{p_0}(\mathbf{T})$) and from $L_{p_1}(\mathbf{T})$ to $L_{p_1}(\mathbf{T})$ ($p_0 \leqslant 1 < p_1 \leqslant \infty$) then it is also bounded from $H_{p,q}(\mathbf{T})$ to $L_{p,q}(\mathbf{T})$ (resp. to $H_{p,q}(\mathbf{T})$) if $p_0 < p < p_1$ and $0 < q \leqslant \infty$.*

For a distribution

$$f \sim \sum_{k=-\infty}^{\infty} \hat{f}(k)\, e^{\imath kx}$$

the *conjugate distribution* is defined by

$$\tilde{f} \sim \sum_{k=-\infty}^{\infty} (-\imath \operatorname{sign} k)\, \hat{f}(k)\, e^{\imath kx}.$$

As is well known, if f is an integrable function then

$$\tilde{f}(x) = \text{p.v.}\, \frac{1}{\pi} \int_{\mathbf{T}} \frac{f(x-t)}{2\tan(t/2)}\, dt := \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{\varepsilon < |t| < \pi} \frac{f(x-t)}{2\tan(t/2)}\, dt.$$

Moreover, the conjugate function $\tilde{f}$ does exist almost everywhere, but it is not integrable in general. It is easy to see that $(\tilde{f})^{\sim} = -f$.

Fefferman and Stein [5] verified that

$$\|f\|_{H_p(\mathbf{T})} \sim \|f\|_p + \|\tilde{f}\|_p \qquad (0 < p < \infty). \tag{3}$$

3. θ-SUMMABILITY OF FOURIER SERIES

First we introduce the Fourier transform for an integrable function $f \in L_1(\mathbf{R})$ by

$$\hat{f}(u) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} f(x)\, e^{-\imath ux}\, dx \qquad (u \in \mathbf{R}).$$

The θ-summation was considered in Butzer and Nessel [3] and, more recently Bokor, Schipp, Szili and Vértesi [2, 11, 12, 16, 17] investigated the uniform convergence of the θ-means and some interpolation problems for continuous functions.

In what follows *we suppose that* $\theta \in L_1(\mathbf{R})$ *is an even continuous function satisfying* $\theta(0)=1$, $\hat{\theta} \in L_1(\mathbf{R})$ *and* $\theta(\frac{\cdot}{n+1}) \in l_1$. Note that this last condition is satisfied if θ is non-increasing on $\mathbf{R}_+$ or if it has compact support.

Denote by $s_n f$ the nth partial sum of the Fourier series of a distribution f, namely,

$$s_n f(x) := \sum_{k=-n}^{n} \hat{f}(k)\, e^{\imath kx}.$$

The θ-*means* of a distribution f are defined by

$$U_n^\theta f(x) := \sum_{k=-\infty}^{\infty} \theta\left(\frac{k}{n+1}\right) \hat{f}(k)\, e^{\imath kx} = (f * K_n^\theta)(x) \qquad (x \in \mathbf{T}), \tag{4}$$

where the K_n^θ kernels satisfy

$$K_n^\theta(t) := \sum_{k=-\infty}^{\infty} \theta\left(\frac{k}{n+1}\right) e^{\imath kt} = 1 + 2 \sum_{k=1}^{\infty} \theta\left(\frac{k}{n+1}\right) \cos(kt) \qquad (t \in \mathbf{T}).$$

It is easy to see that if θ has bounded variation then the θ-summation is permanent, i.e. if $s_n f$ is convergent in some sense then $U_n^\theta f$ is also convergent and converges to the same limit.

Following Butzer and Nessel [3] and Schipp and Bokor [11] we verify a new characterization for the θ-means. We write $U_n^\theta f$ as a singular integral of f and the Fourier transform of θ over the whole real line.

LEMMA 1. *If* $f \in L_1(\mathbf{T})$ *then*

$$U_n^\theta f(x) = (n+1) \int_{-\infty}^{\infty} f(x-t)\, \hat{\theta}((n+1)\, t)\, dt \qquad (n \in \mathbf{N}). \tag{5}$$

Proof. If $f(t) = e^{\imath kt}$ then

$$(n+1) \int_{-\infty}^{\infty} e^{\imath k(x-t)} \hat{\theta}((n+1)\, t)\, dt = e^{\imath kx} \int_{-\infty}^{\infty} e^{-\imath kt/(n+1)} \hat{\theta}(t)\, dt$$

$$= \theta\left(\frac{k}{n+1}\right) e^{\imath kx} = U_n^\theta f(x).$$

Hence the lemma holds also for trigonometric polynomials. Let f be an arbitrary element from $L_1(\mathbf{T})$ and (f_k) be a sequence of trigonometric polynomials such that $f_k \to f$ in $L_1(\mathbf{T})$ norm. The condition $\theta(\frac{\cdot}{n+1}) \in l_1$ implies that $K_n^\theta \in L_1(\mathbf{T})$. Since

$$U_n^\theta f(x) = \frac{1}{2\pi} \int_{\mathbf{T}} f(x-t)\, K_n^\theta(t)\, dt$$

for $f \in L_1(\mathbf{T})$, we can conclude that $U_n^\theta f_k \to U_n^\theta f$ in $L_1(\mathbf{T})$ norm as $k \to \infty$. On the other hand, $\hat{\theta} \in L_1(\mathbf{R})$, and so

$$(n+1) \int_{-\infty}^{\infty} f_k(x-t)\, \hat{\theta}((n+1)\, t)\, dt \to (n+1) \int_{-\infty}^{\infty} f(x-t)\, \hat{\theta}((n+1)\, t)\, dt$$

in $L_1(\mathbf{T})$ norm as $k \to \infty$. This finishes the proof of the lemma. ∎

The *conjugate θ-means* of a distribution f are introduced by

$$\tilde{U}_n^\theta f(x) := \sum_{k=-\infty}^{\infty} \theta\left(\frac{k}{n+1}\right) (-\imath \operatorname{sign} k)\, \hat{f}(k)\, e^{\imath kx}.$$

The *maximal* and *conjugate maximal θ-operators* are defined by

$$U_*^\theta f := \sup_{n \in \mathbf{N}} |U_n^\theta f| \qquad \text{and} \qquad \tilde{U}_*^\theta f := \sup_{n \in \mathbf{N}} |\tilde{U}_n^\theta f|,$$

respectively. Our first boundedness result is the following

LEMMA 2. *The operator U_*^θ is bounded on $L_\infty(\mathbf{T})$.*

Proof. The characterization (5) implies that

$$\| U_n^\theta f \|_\infty \leqslant \| f \|_\infty\, \| \hat{\theta} \|_1$$

for all $n \in \mathbf{N}$, which shows the lemma. ∎

In this paper the constants C are depending only on θ and the constants C_p (resp. $C_{p,q}$) are depending only on p and θ (resp. p, q and θ) and may denote different constants in different contexts.

4. THE BOUNDEDNESS OF THE MAXIMAL θ-OPERATOR

A *generalized interval* on $\mathbf{T}$ is either an interval $I \subset \mathbf{T}$ or $I = [-\pi, x) \cup [y, \pi)$. A bounded measurable function a is a *p-atom* if there exists a generalized interval I such that

(i) $\int_I a(x)\, x^j\, dx = 0$ where $j \in \mathbf{N}$ and $j \leqslant [1/p - 1]$, the integer part of $(1/p - 1)$,

(ii) $\|a\|_\infty \leqslant |I|^{-1/p}$,

(iii) $\{a \neq 0\} \subset I$.

We will use the following two theorems, the first one can be found in Weisz [21].

THEOREM B. *Suppose that the operator V is sublinear and, for some $0 < p \leqslant 1$,*

$$\int_{\mathbf{T} \setminus 8I} |Va|^p \, d\lambda \leqslant C_p \tag{6}$$

for every p-atom a where I is the support of the atom and 8I is the generalized interval with the same center as I and with length $8\,|I|$. If V is bounded from $L_{p_1}(\mathbf{T})$ to $L_{p_1}(\mathbf{T})$ for a fixed $1 < p_1 \leqslant \infty$ then

$$\|Vf\|_p \leqslant C_p \, \|f\|_{H_p(\mathbf{T})} \qquad (f \in H_p(\mathbf{T})).$$

We formulate also a weak version of this theorem, which is an easy modification of a result in Long [8], so we sketch the proof, only.

THEOREM C. *Suppose that the operator V is sublinear and, for some $0 < p < 1$,*

$$\sup_{\rho > 0} \rho^p \, \lambda(\{|Va| > \rho\} \cap \{\mathbf{T} \setminus 8I\}) \leqslant C_p \tag{7}$$

for every p-atom a where I denotes again the support of the atom. If V is bounded from $L_{p_1}(\mathbf{T})$ to $L_{p_1}(\mathbf{T})$ for a fixed $1 < p_1 \leqslant \infty$ then

$$\|Vf\|_{p,\infty} \leqslant C_p \, \|f\|_{H_p(\mathbf{T})} \qquad (f \in H_p(\mathbf{T})).$$

Proof. If (7) is satisfied without the intersection with $\{\mathbf{T} \setminus 8I\}$, then the result can be found in Long [8, p. 279]. Then

$$\sup_{\rho > 0} \rho^p(\{|Va| > \rho\} \cap \{8I\})$$

$$\leqslant \int_{8I} |Va|^p \, d\lambda \leqslant C_p \left(\int_{\mathbf{T}} |Va|^{p_1} \, d\lambda \right)^{p/p_1} |I|^{1 - p/p_1} \leqslant C_p,$$

which proves the theorem. ∎

Now we are ready to prove the boundedness of the maximal operator on the Hardy spaces. First we recall a known result, which was shown in another context. Taking into account (5) we can see that Torchinsky [18, p. 82–84] has proved essentially the next inequality.

PROPOSITION 1. *Assume that there is an even, on* $\mathbf{R}_+$ *non-increasing function* η_0 *such that* $|\hat{\theta}| \leqslant \eta_0$. *If* $\eta_0 \in L_1(\mathbf{R})$ *then*

$$\sup_{\rho>0} \rho\lambda(U_*^\theta f > \rho) \leqslant C \|f\|_1 \qquad (f \in L_1(\mathbf{T})). \tag{8}$$

It follows from Proposition 1 and Lemma 2 and by interpolation that

$$\|U_*^\theta f\|_p \leqslant C \|f\|_p \qquad (f \in L_p(\mathbf{T}),\ 1 < p \leqslant \infty).$$

If we suppose a little bit more on η_0 then we can prove that U_*^θ is bounded from $H_1(\mathbf{T})$ to $L_1(\mathbf{T})$.

THEOREM 1. *Assume that there is an even, on* $\mathbf{R}_+$ *non-increasing function* η_0 *such that* $|\hat{\theta}| \leqslant \eta_0$, $t\eta_0(t)$ *is non-increasing on the interval* $[1, \infty)$. *If* θ *has compact support that is contained in* $[-c, c]$ *and if* $\eta_0 \in L_1(\mathbf{R})$ *then*

$$\|{}_*^\theta f\|_1 \leqslant cC \|f\|_{H_1(\mathbf{T})} \qquad (f \in H_1(\mathbf{T})).$$

Proof. We will verify (6) for $p=1$. Then Theorem 1 will follow from Lemma 2 and Theorem B.

To this end let a be an arbitrary 1-atom with support I and $2^{-K-1} < |I|/\pi \leqslant 2^{-K}$ $(K \in \mathbf{N})$. If t_0 is the center of I, then the center of $I' := I - t_0$ is zero. By changing the variables we can see that

$$\begin{aligned}
\int_{\mathbf{T}\setminus 8I} |U_*^\theta a|^p \, d\lambda &= \int_{\mathbf{T}\setminus 8I} \sup_{n\in\mathbf{N}} \left| \int_I a(t)\, K_n^\theta(x-t)\, dt \right|^p dx \\
&= \int_{\mathbf{T}\setminus 8I'} \sup_{n\in\mathbf{N}} \left| \int_{I'} a'(t)\, K_n^\theta(x-t)\, dt \right|^p dx \\
&= \int_{\mathbf{T}\setminus 8I'} |U_*^\theta a'|^p \, d\lambda,
\end{aligned}$$

where $a'(t) := a(t+t_0)$.

Hence we can suppose that the center of I is zero. In this case

$$[-\pi 2^{-K-2}, \pi 2^{-K-2}] \subset I \subset [-\pi 2^{-K-1}, \pi 2^{-K-1}].$$

First suppose that $x \geqslant 0$. Then

$$\int_{\{\mathbf{T}\backslash 8I\} \cap \{x \geqslant 0\}} |U_*^\theta a(x)| \, dx \leqslant \sum_{i=2}^{2^K-1} \int_{\pi i 2^{-K}}^{\pi(i+1)\, 2^{-K}} \sup_{n+1 \geqslant r_i} |U_n^\theta a(x)| \, dx + \sum_{i=2}^{2^K-1} \int_{\pi i 2^{-K}}^{\pi(i+1)\, 2^{-K}} \sup_{n+1 < r_i} |U_n^\theta a(x)| \, dx = (A) + (B),$$

where $r_i := 2^K i^{-\alpha}$ $(i \in \mathbf{N})$ with $\alpha > 0$ chosen later.

By Lemma 1 and by the condition $|\hat{\theta}| \leqslant \eta_0$ we estimate $U_n^\theta a$ as follows:

$$|U_n^\theta a(x)| \leqslant (n+1) \int_{-\infty}^{\infty} |a(t)| \, \eta_0((n+1)(x-t)) \, dt.$$

Then, by the definition of the 1-atom,

$$\sup_{n+1 \geqslant r_i} |U_n^\theta a(x)| \leqslant 2^K r_i \sum_{k=-\infty}^{\infty} \int_{I+2k\pi} \eta_0(r_i(x-t)) \, dt := (A_1)(x) + (A_2)(x),$$

where (A_1) denotes the term $k=0$ and (A_2) the sum $\sum_{|k|=1}^{\infty}$.

If $t \in I$ and $x \in [\pi i 2^{-K}, \pi(i+1)\, 2^{-K})$ for some $i = 2, ..., 2^K - 1$, then

$$|x-t| \geqslant \pi i 2^{-K} - \pi 2^{-K-1} \geqslant \pi(i-1)\, 2^{-K}. \tag{9}$$

This implies

$$(A_1)(x) \leqslant C 2^K i^{-\alpha} \eta_0(i^{1-\alpha}\pi/2) \qquad (x \in [\pi i 2^{-K}, \pi(i+1)\, 2^{-K})).$$

If $t \in I + 2k\pi$ for some $k \neq 0$ then $|x-t| \sim 2\, |k|\, \pi$. We have

$$(A_2)(x) \leqslant C 2^K i^{-\alpha} \sum_{k=1}^{\infty} \eta_0(2^{K+1} i^{-\alpha} \pi k) \leqslant C \int_0^\infty \eta_0 \, d\lambda \leqslant C.$$

Hence, in case $0 < \alpha < 1$,

$$(A)(x) \leqslant C + C 2^{-K} \sum_{i=2}^{2^K-1} 2^K i^{-\alpha} \eta_0(i^{1-\alpha}\pi/2) \leqslant C \int_0^\infty \eta_0 \, d\lambda \leqslant C.$$

Now let us consider (B). Since $\operatorname{supp} \theta \subset [-c, c]$ and θ is bounded, (4) implies

$$|U_n a(x)| \leqslant C \sum_{|k|=0}^{c(n+1)} |\hat{a}(k)|.$$

As

$$|\hat{a}(k)| = \left| \frac{1}{2\pi} \int_I a(x)(e^{-\imath kx} - 1)\, dx \right| \leqslant C \int_I |a(x)|\, |kx|\, dx \leqslant C\, |k|\, |I|$$

we obtain

$$\sup_{n+1<r_i} |U_n a(x)| \leqslant cCr_i^2 2^{-K} \leqslant cC2^K i^{-2\alpha}.$$

Therefore,

$$(B) \leqslant cC2^{-K} \sum_{i=2}^{2^K-1} 2^K i^{-2\alpha}$$

which is bounded if $1/2 < \alpha < 1$. If $x < 0$ then

$$\int_{\{\mathbf{T}\setminus 8I\} \cap \{x<0\}} |U_*^\theta a(x)|\, dx \leqslant \sum_{i=-2}^{-(2^K-1)} \int_{\pi i 2^{-K}}^{\pi(i-1)\,2^{-K}} \sup_{n+1 \geqslant r_i} |U_n^\theta a(x)|\, dx$$
$$+ \sum_{i=-2}^{-(2^K-1)} \int_{\pi i 2^{-K}}^{\pi(i-1)\,2^{-K}} \sup_{n+1<r_i} |U_n^\theta a(x)|\, dx,$$

where $r_i := 2^K\, |i|^{-\alpha}$. The inequality

$$\int_{\{\mathbf{T}\setminus 8I\} \cap \{x<0\}} |U_*^\theta a(x)|\, dx \leqslant cC$$

can be proved exactly as above. The proof of the theorem is complete. ∎

Remark. We can extend this result to $p < 1$ as follows. In addition to the conditions of Theorem 1 suppose that

$$\int_0^\infty t^{(1-p_0)(1+\varepsilon)/(2p_0-1)}\, \eta_0(t)^{p_0}\, dt < \infty \tag{10}$$

for some $1/2 < p_0 < 1$ and $\varepsilon > 0$. Then we can prove in the same way that

$$\| U_*^\theta f \|_{p_0} \leqslant C_{p_0}\, \|f\|_{H_{p_0}(\mathbf{T})} \qquad (f \in H_{p_0}(\mathbf{T}))$$

and

$$\| U_*^\theta f \|_{p,q} \leqslant C_{p,q}\, \|f\|_{H_{p,q}(\mathbf{T})} \qquad (f \in H_{p,q}(\mathbf{T}))$$

for every $p_0 < p < \infty$ and $0 < q \leqslant \infty$.

If $p_0=1$ then condition (10) reduces to the integrability of η_0. Since $\hat{\theta}$ is bounded, we may suppose that η_0 is also bounded. It is easy to see that if (10) is satisfied for p_0 then it holds also for all $p_0 \leqslant p \leqslant 1$. The interval $[1, \infty)$ in Theorem 1 is a technical condition, only, we could change it to $[c, \infty)$ for any $c>0$.

If we have some information about the derivatives of $\hat{\theta}$ we can prove an even sharper result. Let $\hat{\theta}^{(k)}$ be denote the kth derivative of $\hat{\theta}$.

THEOREM 2. *Assume that there are two even, on $\mathbf{R}_+$ non-increasing functions η_N and η_{N+1} such that $|\hat{\theta}^{(N)}| \leqslant \eta_N$, $0 \neq |\hat{\theta}^{(N+1)}| \leqslant \eta_{N+1}$, $t^{N+1}\eta_N(t)$ is non-increasing on $[1, \infty)$ and $t^{N+2}\eta_{N+1}(t)$ is non-decreasing on $\mathbf{R}_+$ ($N \in \mathbf{N}$). If $\eta_N, \eta_{N+1} \in L_{p_0}(\mathbf{R})$ for some $p_0 \leqslant 1/(N+1)$ then*

$$\| U_*^\theta f \|_{p_0} \leqslant C_{p_0} \| f \|_{H_{p_0}(\mathbf{T})} \qquad (f \in H_{p_0}(\mathbf{T})) \tag{11}$$

and

$$\| U_*^\theta f \|_{p,q} \leqslant C_{p,q} \| f \|_{H_{p,q}(\mathbf{T})} \qquad (f \in H_{p,q}(\mathbf{T})) \tag{12}$$

for every $p_0 < p < \infty$, $0 < q \leqslant \infty$. In particular, if $f \in L_1(\mathbf{T})$ then (8) holds. Moreover, if $\eta_N, \eta_{N+1} \in L_{p_0,\infty}(\mathbf{R})$ for some $p_0 \leqslant 1/(N+1)$ ($p_0 \neq 1$) then

$$\| U_*^\theta f \|_{p_0,\infty} \leqslant C_{p_0} \| f \|_{H_{p_0}(\mathbf{T})} \qquad (f \in H_{p_0}(\mathbf{T})) \tag{13}$$

and (12) and (8) are valid.

Proof. First we show (6) for $p=p_0$. Let a be an arbitrary p_0-atom with support I and $2^{-K-1} < |I|/\pi \leqslant 2^{-K}$ ($K \in \mathbf{N}$). As in the proof of Theorem 1 we can suppose that the center of I is zero. Let $A_0(x) := a(x)$ ($x \in \mathbf{R}$) and

$$A_j(x) := \int_{-\infty}^{x} A_{j-1}(t)\, dt \qquad (x \in \mathbf{R};\ j=1, ..., [1/p-1]+1).$$

By (i) of the definition of the atom we can show that $\operatorname{supp} A_j \subset \bigcup_{k=-\infty}^{\infty} \{I+2k\pi\}$ ($j=1, ..., [1/p-1]+1$). Moreover, by (ii),

$$\|A_j\|_\infty \leqslant |I|^{-1/p+j} \qquad (j=1, ..., [1/p-1]+1). \tag{14}$$

Using Lemma 1 and integrating by parts we can see that

$$|U_n^\theta a(x)| = (n+1)^{N+1} \left| \int_{-\infty}^{\infty} A_N(t)\, \hat{\theta}^{(N)}((n+1)(x-t))\, dt \right|$$

$$\leqslant (n+1)^{N+1} \int_{-\infty}^{\infty} |A_N(t)|\, \eta_N((n+1)(x-t))\, dt. \tag{15}$$

By the conditions of the theorem and (14),

$$\sup_{n+1\geqslant 2^K} |U_n^\theta a(x)| \leqslant 2^{K/p_0+K} \sum_{k=-\infty}^{\infty} \int_{I+2k\pi} \eta_N(2^K(x-t))\,dt$$
$$= (C)(x)+(D)(x),$$

where (C) denotes the term $k=0$ and (B) the sum $\sum_{|k|=1}^{\infty}$.

We suppose again that $x\in[-\pi,\pi)\backslash 8I$ and $x\geqslant 0$. If $t\in I$ and $x\in[\pi i2^{-K}, \pi(i+1)\,2^{-K})$ $(i=2, ..., 2^K-1)$, then (9) implies

$$(C)(x)\leqslant C2^{K/p_0}\eta_{N+1}((i-1)\,\pi) \qquad (x\in[\pi i2^{-K}, \pi(i+1)\,2^{-K})),$$

thus

$$\int_{\{\mathbf{T}\backslash 8I\}\,\cap\,\{x\geqslant 0\}} (C)(x)^{p_0}\,dx \leqslant C_{p_0}2^{-K}\sum_{i=2}^{2^K-1} 2^K\eta_{N+1}^{p_0}((i-1)\,\pi)$$
$$\leqslant C_{p_0}\int_0^\infty \eta_{N+1}^{p_0}\,d\lambda \leqslant C_{p_0}.$$

If $t\in I+2k\pi$ $(k\neq 0)$ then

$$(D)^{p_0}(x)\leqslant C_{p_0}2^K\sum_{k=1}^{\infty}\eta_{N+1}^{p_0}(2^{K+1}\pi k)\leqslant C_{p_0}\int_\varepsilon^\infty \eta_{N+1}^{p_0}\,d\lambda\leqslant C_{p_0}$$

and (6) is satisfied.

Similarly to (15) we can also obtain that

$$|U_n^\theta a(x)|\leqslant (n+1)^{N+2}\int_{-\infty}^{\infty}|A_{N+1}(t)|\,\eta_{N+1}((n+1)(x-t))\,dt$$

and then $\sup_{n+1<2^K}|U_n^\theta a(x)|$ can be estimated in the same way as $\sup_{n+1\geqslant 2^K}|U_n^\theta a(x)|$ above. The case $x<0$ can be treated similarly. This proves inequality (11).

To prove (13) observe that

$$\rho^{p_0}\lambda(\{(C)>\rho\}\cap\{\mathbf{T}\backslash 8I\})=\rho^{p_0}\sum_{i\geqslant 1:\,\eta_{N+1}(i\pi)>\rho 2^{-K/p_0}} 2^{-K}$$
$$\leqslant C_\varepsilon\rho^{p_0}2^{-K}\lambda(\{\eta_{N+1}>\rho 2^{-K/p_0}\})$$
$$\leqslant C_\varepsilon\,\|\eta_{N+1}\|_{L_{p_0,\infty}(\mathbf{R})}^{p_0}.$$

Obviously, (D) satisfies also (7). We can estimate $\sup_{n+1<2^K} |U_n^\theta a(x)|$ similarly, which shows (13). The inequality (12) follows from Theorem A. Applying (1) and (12) for $p=1$ and $q=\infty$, we conclude

$$\| U_*^\theta f \|_{1,\infty} \leqslant C \| f \|_{H_{1,\infty}} \leqslant C \| f \|_1$$

which shows (8). This finishes the proof of the theorem. ∎

Remark. We can weaken the condition $t^{N+1}\eta_N(t) \searrow$ in Theorem 2 by

$$t^{N+1}\eta_N(t) \leqslant t_0^{N+1}\eta_N(t_0) \qquad (t \geqslant t_0 \geqslant 1).$$

Of course we could replace $t^{N+2}\eta_{N+1}(t) \nearrow$ also by an analogous condition.

In the next theorem we show the boundedness of U_*^θ on Hardy spaces if $t^{N+2}\hat{\theta}^{(N+1)}(t)$ is bounded.

THEOREM 3. *Assume that* $0 \neq |t^{N+2}\hat{\theta}^{(N+1)}(t)| \leqslant C$ *for some* $N \in \mathbf{N}$. *Then* (12) *holds for every* $1/(N+2) < p < \infty$, $0 < q \leqslant \infty$ *and*

$$\| U_*^\theta f \|_{1/(N+2),\infty} \leqslant C_{1/(N+2)} \| f \|_{H_{1/(N+2)}(\mathbf{T})} \qquad (f \in H_{1/(N+2)}(\mathbf{T})). \quad (16)$$

Especially, if $f \in L_1(\mathbf{T})$ *then the weak type* (1, 1) *inequality* (8) *holds.*

Proof. First we show (12) for $1/(N+2) < p = q \leqslant 1/(N+1)$. The general case of (12) will follow from Theorem A. To this end, by Lemma 1 and Theorem B we have only to prove that condition (6) is satisfied for $1/(N+2) < p \leqslant 1/(N+1)$. Note that in this case $[1/p - 1] = N$.

Let a be an arbitrary p-atom with support I and $2^{-K-1} < |I|/\pi \leqslant 2^{-K}$ $(K \in \mathbf{N})$. We can suppose again that the center of I is zero. As in (15) we conclude

$$\begin{aligned} |U_n^\theta a(x)| &= (n+1)^{N+2} \left| \int_{-\infty}^{\infty} A_{N+1}(t)\, \hat{\theta}^{(N+1)}((n+1)(x-t))\, dt \right| \\ &\leqslant |I|^{-1/p+N+1} \sum_{k=-\infty}^{\infty} \int_{I+2k\pi} |x-t|^{-(N+2)}\, dt \\ &:= (E)(x) + (F)(x), \end{aligned}$$

where (E) denotes again the term corresponding to $k=0$ and (F) the sum $\sum_{|k|=1}^{\infty}$.

If $t \in I$ and $x \in [\pi i 2^{-K}, \pi(i+1)\, 2^{-K})$ for some $i = 2, ..., 2^K - 1$, then (9) implies

$$(E)(x) \leqslant C 2^{K/p} i^{-(N+2)} \qquad (x \in [\pi i 2^{-K}, \pi(i+1)\, 2^{-K})) \quad (17)$$

and so

$$\int_{\{\mathbf{T}\setminus 8I\}\cap\{x\geqslant 0\}} (E)(x)^p\,dx \leqslant \sum_{i=2}^{2^K-1} \int_{\pi i 2^{-K}}^{\pi(i+1)\,2^{-K}} (E)(x)^p\,dx$$

$$\leqslant C_p 2^{-K} \sum_{i=2}^{2^K-1} 2^K i^{-(N+2)\,p} \leqslant C_p.$$

If $t \in I + 2k\pi$ for some $k \neq 0$ then

$$(F)(x) \leqslant C 2^{-K(-1/p+N+2)} \sum_{k=1}^{\infty} k^{-(N+2)} \leqslant C$$

and (6) is satisfied automatically. If $x<0$ then we can show (6) in the same way.

To prove (16) we have to check (7) for $p=1/(N+2)$. Inequality (17) implies that

$$\rho^{1/(N+2)}\lambda(\{(E)>\rho\}\cap\{\mathbf{T}\setminus 8I\}) = \rho^{1/(N+2)} \sum_{i=1}^{(2^{K(N+2)}\rho-1)^{1/(N+2)}} 2^{-K} \leqslant 1.$$

Since (F) satisfies also (7), we have shown (16). Inequality (8) can be verified by interpolation as in Theorem 2. The proof of the theorem is complete. ▮

Notice that by interpolation we get the inequality

$$\|f\|_{H_{p,q}(\mathbf{T})} \sim \|f\|_{p,q} + \|\tilde{f}\|_{p,q} \qquad (0<p<\infty, 0<q\leqslant\infty)$$

from (3). Since $\|f\|_{H_{p,q}} \sim \|\tilde{f}\|_{H_{p,q}}$ and $\tilde{U}_n^\theta f = U_n^\theta \tilde{f}$, we can extend Theorems 2 and 3 easily to the conjugate maximal operators and to the θ-means as follows.

THEOREM 4. *Theorems 2 and 3 hold also for the operator $\tilde{U}_*^\theta$ instead of U_*^θ. Moreover, if we replace on the left hand side of the inequalities* (11)–(13) *and* (16) *the operator U_*^θ by U_n^θ or $\tilde{U}_n^\theta$ and the space $L_{p,q}$ by $H_{p,q}$, then these inequalities hold uniformly in n.*

Since the trigonometric polynomials are dense in $L_1(\mathbf{T})$ and in the Hardy spaces, the inequalities of Theorems 2-4 and the usual density argument (see Marcinkiewicz, Zygmund [9]) imply

COROLLARY 1. *Under the conditions of Theorem 2 or 3, $f \in L_1(\mathbf{T})$ implies*

$$U_n^\theta f \to f \text{ a.e.} \qquad \text{and} \qquad \tilde{U}_n^\theta f \to \tilde{f} \text{ a.e.} \qquad \text{as} \quad n\to\infty.$$

Moreover, if e.g. (12) *is satisfied, then these two convergence hold in* $H_{p,q}$ *norm, whenever* $f \in H_{p,q}$, *if* (13) *is true, then in* $H_{p_0,\infty}$ *norm, whenever* $f \in H_{p_0}$. *From Theorems* 3 *and* 4 *we obtain similar convergence results, i.e. if* (16) *is satisfied.*

Note that $\tilde{f}$ is not necessarily integrable whenever f is.

5. APPLICATIONS TO VARIOUS SUMMABILITY METHODS

In this section we consider several summability methods introduced in the book of Butzer and Nessel [3] and some other popular ones as special cases of the θ-summation. Of course, there are a lot of other summability methods which could be considered as special cases. The elementary computations in the examples below are left to the reader.

Let C_0 consists of all continuous functions f, for which $\lim_{|x|\to\infty} f(x) = 0$. Butzer and Nessel [3] verified that if $\theta \in C_0$ and θ, θ' and $x\theta''(x)$ are integrable functions, then $\hat{\theta} \in L_1(\mathbf{R})$.

LEMMA 3. *Suppose that* $\theta \in L_1(\mathbf{R}) \cap C_0$ *is even and each term of* $(x^i\theta(x))^{(i+1)}$ *is integrable for some* $i \geqslant 0$. *Then* $\hat{\theta} \in L_1(\mathbf{R})$ *and*

$$|\hat{\theta}^{(i)}(x)| \leqslant \frac{C}{x^{i+1}} \qquad (x \in (0, \infty)).$$

Proof. The integrability of $\hat{\theta}$ comes from the result above. By integrating by parts we have

$$|\hat{\theta}^{(i)}(x)| = \left| \int_0^\infty t^i\theta(t) \cos tx \, dt \right| = \frac{1}{x} \left| \int_0^\infty (t^i\theta(t))' \sin tx \, dt \right| = \cdots$$

$$= \frac{1}{x^{i+1}} \left| [\theta(t) \cos tx]_0^\infty \right| + \frac{1}{x^{i+1}} \left| \int_0^\infty (t^i\theta(t))^{(i+1)} \cos tx \, dt \right|.$$

Of course, in the last line probably cos have to changed to sin. ∎

Our first three examples satisfy the conditions of Lemma 3.

EXAMPLE 1. *Weierstrass summation.* Let $\theta_1(x) = e^{-|x|^\gamma}$ for some $0 < \gamma < \infty$. It is easy to see that $(x^i e^{-|x|^\gamma})^{(i+1)} \in L_1(\mathbf{R})$ for all $i \geqslant 0$. The θ-means are given by

$$U_n^{\theta_1} f(x) := \sum_{k=-\infty}^{\infty} e^{-(|k|/(n+1))^\gamma} \hat{f}(k)\, e^{\imath kx}.$$

Of course, we can take another index set than $\mathbf{N}$. For example we can change $(\frac{1}{n+1})^\gamma$ to t:

$$V_t^{\theta_1} f(x) := \sum_{k=-\infty}^{\infty} e^{-t\,|k|^\gamma} \hat{f}(k)\, e^{\imath kx} \qquad (t \in (0, \infty)),$$

or e^{-t} by r:

$$W_r^{\theta_1} f(x) := \sum_{k=-\infty}^{\infty} r^{|k|^\gamma} \hat{f}(k)\, e^{\imath kx} \qquad (r \in (0, 1)).$$

By Lemma 3, θ_1 satisfies the conditions of Theorem 3 for all $N \in \mathbf{N}$. This means e.g. that the operators $U_*^{\theta_1}$, $V_*^{\theta_1}$ and $W_*^{\theta_1}$ are bounded from $H_{p,q}(\mathbf{T})$ to $L_{p,q}(\mathbf{T})$ for every $0<p<\infty$ and $0<q\leqslant\infty$. Moreover, $U_n^{\theta_1} f \to f$ a.e. as $n\to\infty$, $V_t^{\theta_1} f\to f$ a.e. as $t\to 0$ and $W_r^{\theta_1} f \to f$ a.e. as $r\to 1$. If $\gamma=1$ then this last result is the well known convergence of the Abel means.

EXAMPLE 2. *Picar summation.* Let $\theta_2(x)=(1+|x|^\gamma)^{-1}$ for some $1<\gamma<\infty$. One can check that $(x^i(1+|x|^\gamma)^{-1})^{(i+1)} \in L_1(\mathbf{R})$ for all $i\geqslant 0$. The θ-means are given by

$$U_n^{\theta_2} f(x) := \sum_{k=-\infty}^{\infty} \frac{1}{1+\left(\frac{|k|}{n+1}\right)^\theta} \hat{f}(k)\, e^{\imath kx}.$$

It follows from Lemma 3 that Theorems 3 and 4 and Corollary 1 hold for this summability method. For example, $U_*^{\theta_2}$ is bounded from $H_{p,q}(\mathbf{T})$ to $L_{p,q}(\mathbf{T})$ for every $0<p<\infty$ and $0<q\leqslant\infty$.

EXAMPLE 3. *Bessel assumption.* Let $\theta_3(x)=(1+x^2)^{-\gamma/2}$ for some $1<\gamma<\infty$. Again, $(x^i(1+x^2)^{-\gamma/2})^{(i+1)}\in L_1(\mathbf{R})$ for all $i\geqslant 0$. The θ-means are given by

$$U_n^{\theta_3} f(x) := \sum_{k=-\infty}^{\infty} \frac{1}{\left(1+\left(\frac{k}{n+1}\right)^2\right)^{\gamma/2}} \hat{f}(k)\, e^{\imath kx}.$$

Thus Theorems 3 and 4 and Corollary 1 hold again for every $0<p<\infty$ and $0<q\leqslant\infty$.

The next six θ-functions are special cases of Theorem 3.

EXAMPLE 4. Fejér summation. Let

$$\theta_4(x) = \begin{cases} 1-|x| & \text{if} \quad |x| \leqslant 1 \\ 0 & \text{if} \quad |x| > 1. \end{cases}$$

$U_n^{\theta_4}$ is the nth Fejér operator:

$$U_n^{\theta_4} f(x) := \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n+1}\right) \hat{f}(k)\, e^{\imath kx} = \frac{1}{n+1} \sum_{k=0}^{n} s_k f(x).$$

It is know that

$$\hat{\theta}_4(x) = \frac{1}{\sqrt{2\pi}} \left(\frac{\sin x/2}{x/2}\right)^2$$

and $|\hat{\theta}_4'(x)| \leqslant C/x^2$. Consequently, Theorems 3 and 4 and Corollary 1 hold for $N=0$.

EXAMPLE 5. *Riemann summation.* Let

$$\theta_5(x) = \left(\frac{\sin x/2}{x/2}\right)^2 = \sqrt{2\pi}\, \hat{\theta}_4(x).$$

Then $\hat{\theta}_5(x) = \sqrt{2\pi}\, \theta_4(x) = \sqrt{2\pi} \max(0, 1-|x|)$ and so $|\hat{\theta}_5'(x)| = 1_{(-1,1)}(x) \leqslant C/x^2$. The Riemann means are given by

$$U_n^{\theta_5} f(x) := \sum_{k=-\infty}^{\infty} \left(\frac{\sin k/(2(n+1))}{k/(2(n+1))}\right)^2 \hat{f}(k)\, e^{\imath kx}.$$

If we change $1/(n+1)$ to μ then

$$V_\mu^{\theta_5} f(x) := \sum_{k=-\infty}^{\infty} \left(\frac{\sin k\mu/2}{k\mu/2}\right)^2 \hat{f}(k)\, e^{\imath kx} \qquad (\mu \in (0, \infty)).$$

This yields that the operators $U_*^{\theta_5}$ and $V_*^{\theta_5}$ are bounded from $H_{p,q}(\mathbf{T})$ to $L_{p,q}(\mathbf{T})$ for every $1/2 < p < \infty$ and $0 < q \leqslant \infty$. Moreover, $U_n^{\theta_5} \to f$ a.e. as $n \to \infty$ and $V_\mu^{\theta_5} \to f$ a.e. as $\mu \to 0$. Note that the Riemann summation was considered in Bari [1], Zygmund [23], Gevorkyan [6, 7] and also in Weisz [20].

EXAMPLE 6. *de La Valleé-Poussin summation.* Let

$$\theta_6(x) = \begin{cases} 1 & \text{if} \quad |x| \leqslant 1/2 \\ -2\,|x| + 2 & \text{if} \quad 1/2 < |x| \leqslant 1 \\ 0 & \text{if} \quad |x| > 1. \end{cases}$$

One can show that $U^{\theta_6}_{2n+1} f = 2U^{\theta_4}_{2n+1} f - U^{\theta_4}_{n} f$ and since $\theta_6(x) = 2\theta_4(x) - \theta_4(2x)$, we have $|\hat{\theta}'_6(x)| \leqslant C/x^2$ (cf. Schipp and Bokor [11]). Hence we get again Theorems 3 and 4 and Corollary 1 for $N=0$. Note that we could generalize this summation if we take in the definition of θ_6 another number than 1/2.

EXAMPLE 7. *Rogosinski summation.* Let

$$\theta_7(x) = \begin{cases} \cos \pi x/2 & \text{if} \quad |x| \leqslant 1 \\ 0 & \text{if} \quad |x| > 1 \end{cases}$$

and

$$U^{\theta_7}_n f(x) := \sum_{k=-n}^{n} \cos\left(\frac{\pi\,|k|}{2(n+1)}\right) \hat{f}(k)\, e^{\imath kx}.$$

Since

$$\hat{\theta}_7(x) = \frac{\sin(x - \pi/2)}{2(x^2 - (\pi/2)^2)}$$

(see e.g. Schipp and Bokor [11]), we can verify that $|\hat{\theta}'_7(x)| \leqslant C/x^2$ and so we obtain Theorems 3 and 4 and Corollary 1 for $N=0$.

EXAMPLE 8. *Jackson-de La Valleé-Poussin summation.* Let

$$\theta_8(x) = \begin{cases} 1 - 3x^2/2 + 3\,|x|^3/4 & \text{if} \quad |x| \leqslant 1 \\ (2 - |x|)^3/4 & \text{if} \quad 1 < |x| \leqslant 2 \\ 0 & \text{if} \quad |x| > 2. \end{cases}$$

One can find in Butzer and Nessel [3] that

$$\hat{\theta}_8(x) = \frac{3}{\sqrt{8\pi}} \left(\frac{\sin x/2}{x/2}\right)^4.$$

Therefore we can show by elementary computations that $|\hat{\theta}^{(i)}_8(x)| \leqslant C/x^4$ for $i = 0,\ 1,\ 2,\ 3$. Consequently, we have the corresponding results with $N=2$.

EXAMPLE 9. *The Summation method of cardinal B-splines.* For $m \geqslant 2$ let

$$M_m(x) := \frac{1}{(m-1)!} \sum_{k=0}^{l} (-1)^k \binom{m}{k} (x-k)^{m-1}$$

$$(x \in [l, l+1),\ l = 0,\ 1,\ ...,\ m-1)$$

and

$$\theta_9(x) = \frac{M_m(m/2 + mx/2)}{M_m(m/2)}.$$

It is shown in Schipp and Bokor [11] that θ_9 is even and

$$\hat{\theta}_9(x) = \frac{1}{\pi m M_m(m/2)} \left(\frac{\sin x/m}{x/m} \right)^m.$$

It is easy to see that $|\hat{\theta}_9^{(i)}(x)| \leqslant C/x^m$ for $i = 0, 1, ..., m-1$. Thus Theorems 3 and 4 and Corollary 1 hold for $N = m-2$.

The next example satisfies the conditions of Theorem 2.

Example 10. *Riesz summation.* Let

$$\theta_{10}(x) = \begin{cases} (1-|x|^\gamma)^\alpha & \text{if} \quad |x| \leqslant 1 \\ 0 & \text{if} \quad |x| > 1 \end{cases}$$

for some $0 < \alpha \leqslant 1 \leqslant \gamma < \infty$. The Riesz operators are given by

$$U_n^{\theta_{10}} f(x) := \sum_{k=-n}^{n} \left(1 - \left| \frac{k}{n+1} \right|^\gamma \right)^\alpha \hat{f}(k) \, e^{\imath kx}.$$

We proved in [21] that

$$|\hat{\theta}_{10}(x)|, \; |\hat{\theta}'_{10}(x)| \leqslant C/x^{\alpha+1}$$

As $0 < \alpha \leqslant 1$, in Theorem 2 we have $N = 0$. It is easy to see that $C/x^{\alpha+1} \in L_{p_0}[\varepsilon, \infty)$ if and only if $p_0 > 1/(\alpha+1)$ and $C/x^{\alpha+1} \in L_{p_0, \infty}[\varepsilon, \infty)$ if and only if $p_0 \geqslant 1/(\alpha+1)$. Consequently, (12) holds for $1/(\alpha+1) < p < \infty$ and (13) for $1/(\alpha+1) \leqslant p_0 < \infty$ and the endpoints are the same in Theorem 4 and Corollary 1.

6. θ-SUMMATION OF FOURIER TRANSFORMS

In this section we summarize briefly the above results for Fourier transforms. First we introduce the Hardy spaces on the real line.

The *Fourier transform* of a tempered distribution f is denoted by $\hat{f}$. The non-tangential maximal function of a tempered distribution is defined by

$$f^*(x) := \sup_{t>0} |(f * P_t)(x)|,$$

where

$$P_t(x) := \frac{ct}{t^2 + x^2} \qquad (t > 0,\ x \in \mathbf{R})$$

is the non-periodic Poisson kernel.

The *Hardy–Lorentz space* $H_{p,q}(\mathbf{R})$ $(0 < p,\ q \leqslant \infty)$ consists of all tempered distributions f for which

$$\|f\|_{H_{p,q}(\mathbf{R})} := \|f^*\|_{p,q} < \infty.$$

For a tempered distribution $f \in H_p(\mathbf{R})$ $(0 < p < \infty)$ the *Hilbert transform* or the *conjugate distribution* $\tilde{f}$ is defined by

$$\tilde{f} := f * \Phi,$$

where

$$\hat{\Phi}(u) = -\imath \operatorname{sign} u, \qquad \Phi(x) = \frac{1}{\pi x}.$$

We remark that the analogues of (1), (2) and (3) and the analogues of Theorem A, B and C are true in this case (cf. Weisz [21] and the references there).

For $f \in L_p(\mathbf{R})$ $(1 \leqslant p \leqslant 2)$ the θ-*means* are defined by

$$U_T^\theta f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \theta\left(\frac{t}{T}\right) \hat{f}(t)\, e^{\imath xt}\, dt = (f * K_T^\theta)(x),$$

where

$$K_T^\theta(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \theta\left(\frac{t}{T}\right) e^{\imath xt}\, dt = T\hat{\theta}(Tx).$$

Thus the θ-means can rewritten as

$$U_T^\theta f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, T\hat{\theta}(T(x-t))\, dt.$$

We extend the definition of the θ-means to tempered distributions as follows:

$$U_T^\theta f := f * K_T^\theta \qquad (T > 0).$$

One can show that $U_T^\theta f$ is well defined for all tempered distributions $f \in H_p(\mathbf{R})$ $(0 < p \leqslant \infty)$ and for all functions $f \in L_p(\mathbf{R})$ $(1 \leqslant p \leqslant \infty)$ (cf. Stein [15]). The definition of the *conjugate θ-means* is

$$\tilde{U}_T^\theta f := \tilde{f} * K_T^\theta \qquad (T > 0).$$

The *maximal* and *conjugate maximal θ-operators* are introduced by

$$U_*^\theta f := \sup_{T>0} |U_T^\theta f| \qquad \text{and} \qquad \tilde{U}_*^\theta f := \sup_{T>0} |\tilde{U}_T^\theta f|,$$

respectively.

We can prove all the results of Section 4 also for tempered distributions and Fourier transforms and for the Hardy spaces $H_{p,q}(\mathbf{R})$. We do not formulate exactly the theorems and proofs, because they are almost the same as in Section 4.

THEOREM 5. *Theorems* 1–4, *Proposition* 1 *and Corollary* 1 *hold also for the operators U^θ acting on tempered distributions and defined in this section and for the Hardy spaces $H_{p,q}(\mathbf{R})$.*

Note that the applications of Section 5 are also special cases of the θ-summation of Fourier transforms, the details are left to the reader.

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